5-1 Angles and Their Measure

Objectives
After studying this lesson, you should be able to:
- change from radian to degree measure, and vice versa,
- find angles that are coterminal with a given angle, and
- find the reference angle for a given angle.

Application
In order to locate every point on Earth, cartographers use a grid that contains circles through the poles, called longitude lines, and circles parallel to the equator, called latitude lines. Point P is located by traveling north from the equator through a central angle of $a^\circ$ to a circle of latitude and then west along that circle through an angle of $b^\circ$. If you consult an atlas, you will see, for example, that Houston, Texas, is located at north latitude $29^\circ 45' 26^\prime$ and west longitude $95^\circ 21' 37^\prime$.

An angle may be generated by the rotation of two rays that share a fixed endpoint. Let one ray remain fixed to form the initial side of the angle, and let the second ray rotate to form the terminal side. If the rotation is in a counterclockwise direction, the angle formed is a positive angle. If the rotation is clockwise, it is a negative angle. An angle with its vertex at the origin and its initial side along the positive x-axis is said to be in standard position. If the terminal side of an angle in standard position coincides with one of the axes, the angle is called a quadrantal angle.

In the figures below, all of the angles are in standard position. The measure of angle $A$ is positive, angle $B$ has a negative measure, and angle $C$ is a quadrantal angle with a positive measure.
The two most common units used to measure angles are degrees and radians. You may be more familiar with degree measure. An angle has a measure of one degree (written $1^\circ$) if it results from $\frac{1}{360}$ of a complete revolution in the positive direction. Each degree is comprised of 60 minutes (written $60'$) and each minute is comprised of 60 seconds (written $60''$). The latitude of Houston, Texas, shown in the application at the beginning of the lesson, would be read "29 degrees, 45 minutes, 26 seconds."

Example 1  
**Change $29^\circ45'26''$ to a decimal number of degrees to the nearest thousandth.**

\[
29^\circ45'26'' = 29^\circ + 45'\left(\frac{1^\circ}{60'}\right) + 26''\left(\frac{1^\circ}{3600''}\right) \\
= 29.757^\circ
\]

Some scientific calculators have keys that allow you to change automatically from degrees, minutes, and seconds to decimal values of degrees, and vice versa. Check your calculator's instruction booklet to see if your calculator does this.

Another unit of angle measure is the radian. The definition of a radian is based on the concept of a unit circle. A unit circle is a circle of radius 1 whose center is at the origin of a rectangular coordinate system. The unit circle is symmetric with respect to the $x$-axis, the $y$-axis, and the origin. A point $P(x, y)$ is on the unit circle if and only if its distance from the origin is 1. Thus, for each point $P(x, y)$ on the unit circle, the distance from the origin is represented by the following equation.

\[
\sqrt{(x - 0)^2 + (y - 0)^2} = 1
\]

If each side of this equation is squared, the result is an equation of the unit circle.

\[
x^2 + y^2 = 1
\]

Consider an angle $\alpha$ in standard position, shown at the right. Let $P(x, y)$ be the point of intersection of its terminal side with the unit circle. The radian measure of an angle in standard position is defined as the length of the corresponding arc on the unit circle. Thus, the measure of angle $\alpha$ is $s$ radians.
There is an important relationship between radian and degree measure. Since an angle of one complete revolution can be represented either by 360° or by \( 2\pi \) radians, \( 360° = 2\pi \) radians. Thus, \( 180° = \pi \) radians, and \( 90° = \frac{\pi}{2} \) radians. The following formulas relate degree and radian measures.

<table>
<thead>
<tr>
<th>Degree/Radian Conversion Formulas</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 radian = ( \frac{180}{\pi} ) degrees or about 57.3°</td>
</tr>
<tr>
<td>1 degree = ( \frac{\pi}{180} ) radians or about 0.017 radians</td>
</tr>
</tbody>
</table>

Angles expressed in radians are often written in terms of \( \pi \). The term radians is also usually omitted when writing angle measures. However, the degree symbol is always used in this book to express the measure of angles in degrees.

**Examples**

2. Change 30° to radian measure in terms of \( \pi \).

\[
30° = 30° \times \frac{\pi}{180°} = \frac{\pi}{6}
\]

3. Change \( \frac{3\pi}{4} \) radians to degree measure.

\[
\frac{3\pi}{4} = \frac{3\pi}{4} \times \frac{180°}{\pi} = 135°
\]

Angles whose measures are multiples of 30° and 45° are commonly used in trigonometry. These angle measures correspond to radian measures of \( \frac{\pi}{6} \) and \( \frac{\pi}{4} \), respectively. The diagrams below can help you to make these conversions mentally.

![Multiples of 30° and \( \frac{\pi}{6} \)](image1)

![Multiples of 45° and \( \frac{\pi}{4} \)](image2)

Two angles in standard position are called coterminal angles if they have the same terminal side. Since angles differing in radian measure by multiples of \( 2\pi \) are equivalent, and angles differing in degree measure by multiples of 360° are equivalent, every angle has infinitely many coterminal angles.
Coterminal Angles

If $\alpha$ is the degree measure of an angle, then all angles of the form $\alpha + 360k^\circ$, where $k$ is an integer, are coterminal with $\alpha$. If $\beta$ is the radian measure of an angle, then all angles of the form $\beta + 2k\pi$, where $k$ is an integer, are coterminal with $\beta$.

$k$ is generally assumed to be nonzero.

Example 4
Find one positive angle and one negative angle that are coterminal with an angle having measure $\frac{11\pi}{4}$.

A positive angle is $\frac{11\pi}{4} - 2\pi$ or $\frac{3\pi}{4}$.

A negative angle is $\frac{11\pi}{4} - 4\pi$ or $\frac{5\pi}{4}$.

Example 5
Identify all angles that are coterminal with a 60° angle.

All angles having a measure $60 + 360k^\circ$, where $k$ is an integer, are coterminal with a 60° angle.

If $\alpha$ is a nonquadrantal angle in standard position, its reference angle is defined as the acute angle formed by the terminal side of the given angle and the x-axis. You can use the figures and the rule below to find the reference angle for any angle $\alpha$ that $0 < \alpha < 2\pi$. If the measure of $\alpha$ is greater than $2\pi$ or less than 0, it can be associated with a coterminal angle of positive measure between 0 and $2\pi$.

Reference Angle Rule

For any angle $\alpha$, $0 < \alpha < 2\pi$, its reference angle $\alpha'$ is defined by
a. $\alpha$, when the terminal side is in Quadrant I,
b. $\pi - \alpha$, when the terminal side is in Quadrant II,
c. $\alpha - \pi$, when the terminal side is in Quadrant III, and
d. $2\pi - \alpha$, when the terminal side is in Quadrant IV.
Example 6

Find the measure of the reference angle for each angle.

a. \( \frac{5\pi}{4} \)

b. \( -\frac{13\pi}{3} \)

c. \( \frac{5\pi}{4} \), so its terminal side is in Quadrant III.

a. \( \alpha = \frac{5\pi}{4} \) is coterminal with \( \frac{5\pi}{3} \) in Quadrant IV.

\[
\alpha' = \alpha - \pi \quad \text{Reference Rule c}
\]
\[
= \frac{5\pi}{4} - \pi
\]
\[
= \frac{\pi}{4}
\]

\[\alpha' = 2\pi - \alpha \quad \text{Reference Rule d}
\]
\[
= 2\pi - \frac{5\pi}{3}
\]
\[
= \frac{\pi}{3}
\]

c. 510°

c. 510° is coterminal with 150° in Quadrant II.

\[
\alpha' = \pi - \alpha \quad \text{Reference Rule b}
\]
\[
= 180° - 150
\]
\[
= 30°
\]

CHECKING FOR UNDERSTANDING

Communicating
Mathematics

Read and study the lesson to answer each question.

1. Define the terms positive angle and negative angle in your own words.

2. Explain how to change the measure of an angle from degrees to radians and from radians to degrees.

3. Sketch a 30° angle in standard position and a positive angle greater than 360° that is coterminal with it.

4. Write an expression for the measures of all angles that are coterminal with an angle whose measure is \( \theta \).

Guided
Practice

If each angle has the given measure and is in standard position, determine the quadrant in which its terminal side lies.

5. \( \frac{15\pi}{4} \)

6. 210°

7. -220°

8. \( \frac{11\pi}{6} \)

9. \( \frac{4\pi}{3} \)

10. 750°

11. \( \frac{14\pi}{3} \)

12. -475°

Change each degree measure to radian measure in terms of π.

13. 18°

14. 240°

15. 1°

16. -45°
Central Angles and Arcs

Objectives
After studying this lesson, you should be able to:
- find the length of an arc, given the measure of the central angle,
- find linear and angular velocities, and
- find the area of a sector.

Application
Civil engineers must deal with angle measures and distances frequently when they design roadways. The map below shows a stretch of Route 3 outside of Boston, Massachusetts. Two portions of the roadway are arcs of circles.

How many miles is it along the roadway from point A to point E? To solve this problem, you need to be familiar with central angles. *This problem will be solved in Example 2.*

A **central angle** of a circle is an angle whose vertex lies at the center of the circle. If two central angles in different circles are congruent, the ratio of the lengths of their intercepted arcs is equal to the ratio of the measures of their radii. For example, given \( \angle O \cong \angle Q \), then

\[
\frac{m\angle A}{m\angle D} = \frac{OA}{QC}
\]
Let $O$ be the center of two concentric circles. Let $r$ be the measure of the radius of the larger circle, and let the smaller circle be a unit circle. A central angle of $\theta$ radians is drawn in the two circles that intercept $RT$ on the unit circle and $SW$ on the other circle. Suppose $SW$ is $s$ units long. $RT$ is $\theta$ units long since it is an arc of a unit circle intercepted by a central angle of $\theta$ radians. Thus, we can write the following proportion.

$$\frac{s}{\theta} = \frac{R}{1} \text{ or } s = r\theta$$

**Length of an Arc**

The length of any circular arc, $s$, is equal to the product of the measure of the radius of the circle, $r$, and the radian measure of the central angle, $\theta$, that it subtends.

$$s = r\theta$$

**Example 1**

Find the length of an arc that subtends a central angle of 42° in a circle of radius 8 cm.

$$42^\circ = 42 \times \frac{\pi}{180} \quad \text{Find the radian measure of the central angle.}$$

$$\approx \frac{7\pi}{30} \quad \text{Find the length of the arc.}$$

$$s = r\theta \approx (8) \left(\frac{7\pi}{30}\right)$$

$$\approx 5.86$$

The length of the arc is approximately 5.86 centimeters.

**Example 2**

Refer to the application at the beginning of the lesson. How far is it along the roadway from point $A$ to point $E$?

To find the length of $BC$, change $90.5^\circ$ to radians.

$$90.5^\circ \approx 1.58 \text{ radians}$$

Thus, the length of $BC$ is about $0.30 \times 1.58$ or 0.47 miles.

$$r = 0.30$$

To find the length of $CD$, change $52.5^\circ$ to radians.

$$52.5^\circ \approx 0.92 \text{ radians}$$

Thus, the length of $CD$ is about $0.50 \times 0.92$, or 0.46 miles.

$$r = 0.50$$

Find the sum of the distances.

$$0.43 + 0.50 + 0.47 + 0.46 = 1.86$$

Therefore, the distance from point $A$ to point $E$ is about 1.86 miles.
The arc length formula can be used to find the relationship between the linear and angular velocities of an object moving in a circular path. If the object moves with constant linear velocity \( \bar{v} \) for a period of time \( t \), the distance \( s \) it travels is given by the formula \( s = \bar{v} \cdot t \). Thus, the linear velocity is \( \bar{v} = \frac{s}{t} \).

As the object moves along the circular path, the radius, \( r \), forms a central angle of measure \( \theta \). The change in the central angle with respect to time, \( \frac{\theta}{t} \), is the angular velocity of the object. Since the length of the arc is \( s = r \theta \), the following is true.

\[
\frac{s}{t} = \frac{r \theta}{t}
\]

Divide each side by \( t \).

\[
\bar{v} = \frac{\theta}{t}
\]

Remember that \( \bar{v} = \frac{s}{t} \).

If an object moves along a circle of radius \( r \) units, then its linear velocity, \( \bar{v} \), is given by

\[
\bar{v} = r \frac{\theta}{t}
\]

where \( \frac{\theta}{t} \) represents the angular velocity in radians per unit of time.

---

**Example 3**

A pulley of radius 12 cm turns at 7 revolutions per second. What is the linear velocity of the belt driving the pulley in meters per second?

\[
\bar{v} = r \frac{\theta}{t}
\]

\[
= 12 \left( \frac{7 \cdot 2\pi}{1} \right)
\]

One revolution is \( 2\pi \), so 7 revolutions is \( 7 \cdot 2\pi \) radians.

\[
= 168\pi
\]

\[
\approx 527.788
\]

\[
\approx 5.278
\]

Divide by 100 to change cm to m.

The linear velocity of the belt is approximately 5.28 m/s.

---

**Example 4**

A trucker drives 55 miles per hour. His truck’s tires have a diameter of 26 inches. What is the angular velocity of the wheels in revolutions per second?

First, convert 55 miles per hour to inches per second.

\[
\frac{55 \text{ miles}}{1 \text{ hour}} \times \frac{5280 \text{ feet}}{1 \text{ mile}} \times \frac{12 \text{ inches}}{1 \text{ foot}} \times \frac{1 \text{ hour}}{3600 \text{ seconds}} = \frac{968 \text{ inches}}{1 \text{ second}}
\]

(continued on the next page)

LESSON 5-2 CENTRAL ANGLES AND ARCS 249
Then use the formula for linear velocity to find $\theta$.

\[ v = r \theta \]

\[ \frac{968}{13} = \theta \]

Since $\theta = \frac{968}{13}$ radians, the angular velocity is $\frac{968}{13} + 2\pi$, or about 12 revolutions per second.

A sector of a circle is a region bounded by a central angle and the intercepted arc. For example, figure $ORS$ is a sector of $\odot O$. The ratio of the area of a sector to the area of a circle is equal to the ratio of its arc length to the circumference. Let $A$ represent the area of the sector. Then, $\frac{A}{\pi r^2} = \frac{\text{length of } RTS}{2\pi r}$.

If $\theta$ is the measure of the central angle expressed in radians and $r$ is the measure of the radius of the circle, then the area of the sector, $A$, is as follows.

\[ A = \frac{1}{2} r^2 \theta \]

**Example 5**

A sector has arc length of 16 cm and a central angle measuring 0.95 radians. Find the radius of the circle and the area of the sector.

First, find the radius of the circle.

\[ s = r\theta \]

\[ 16 = r \cdot 0.95 \]

\[ r = 16.84 \quad \text{The radius is about 16.84 cm long.} \]

Then find the area of the sector.

\[ A = \frac{1}{2} r^2 \theta \]

\[ = \frac{1}{2} (16.84)^2 (0.95) \]

\[ = 134.70 \quad \text{The area is about 134.70 cm}^2. \]
5-3  Circular Functions

Objective  After studying this lesson, you should be able to:

- find the values of the six trigonometric functions of an angle in standard position given a point on its terminal side.

Application  At the scene of many traffic accidents, police officers often use a trundle wheel to determine the length of skid marks. When the wheel is turned through one complete revolution, there is a one-to-one correspondence between the points on the rim of the wheel and the real numbers representing the length of the skid marks on the ground. As the wheel completes more revolutions, there is a many-to-one correspondence between the points on the rim of the wheel and the real numbers.

In a similar manner, two important trigonometric functions, the sine and cosine functions, can be defined in terms of the unit circle.

Consider an angle \( \theta \) standard position. The terminal side of the angle intersects the unit circle at a unique point, \( P(x, y) \). The \( y \)-coordinate of this point is called sine \( \theta \). The abbreviation for sine is \( \sin \). The \( x \)-coordinate of this point is called cosine \( \theta \). The abbreviation for cosine is \( \cos \).

**Definition of Sine and Cosine**

If the terminal side of an angle \( \theta \) in standard position intersects the unit circle at \( P(x, y) \), then \( \cos \theta = x \) and \( \sin \theta = y \).

Since there is exactly one point \( P(x, y) \) for any angle \( \theta \), the relations \( \cos \theta = x \) and \( \sin \theta = y \) are functions of \( \theta \). Because they are both defined using a unit circle, they are often called circular functions.

**Example 1**  Find each value.

a. \( \sin 90^\circ \)

b. The terminal side of a \( 90^\circ \) angle in standard position is the positive \( y \)-axis, which intersects the unit circle at \((0, 1)\). The \( y \)-coordinate of this ordered pair is \( \sin 90^\circ \).

Therefore, \( \sin 90^\circ = 1 \).
b. \( \cos \pi \)

b. The terminal side of an angle in standard position measuring \( \pi \) radians is the negative \( x \)-axis, which intersects the unit circle at \((-1, 0)\). The \( x \)-coordinate of this ordered pair is \( \cos \pi \).

Therefore, \( \cos \pi = -1 \).

---

**EXPLORATION: Graphing Calculator**

1. Use the range values below to set up a viewing window.

   \[
   \begin{align*}
   \text{Tmin} &= 0 \\
   \text{Tmax} &= 360 \\
   \text{Xmin} &= -2.5 \\
   \text{Xmax} &= 2.5 \\
   \text{Ymin} &= -1.5 \\
   \text{Ymax} &= 1.5 \\
   \text{Xscale} &= 0.5 \\
   \text{Yscale} &= 0.5
   \end{align*}
   \]

2. Define the unit circle with the definition \( X_T = \cos T \) and \( Y_T = \sin T \).

3. Activate the TRACE function to move around the circle.

   a. What does \( T \) represent?
   
   b. What does the \( x \)-value represent?
   
   c. What does the \( y \)-value represent?

4. Determine the trigonometric functions of the angles whose terminal sides lie at \( 0^\circ, 90^\circ, 180^\circ, \) and \( 360^\circ \).

---

The sine and cosine functions of an angle in standard position may be defined in terms of the ordered pair for any point on its terminal side and the distance between that point and the origin.

Suppose \( P(x, y) \) and \( P'(x', y') \) are two points on the terminal side of an angle with measure \( \theta \), where \( P' \) is on the unit circle. Let \( OP = r \). By the Pythagorean theorem, \( r = \sqrt{x^2 + y^2} \). Since \( P' \) is on the unit circle, \( OP' = 1 \). Triangles \( OPQ \) and \( OPQ' \) are similar. Thus, the lengths of corresponding sides are proportional.

\[
\frac{x'}{1} = \frac{x}{r} \quad \text{and} \quad \frac{y'}{1} = \frac{y}{r}
\]

Therefore, \( \cos \theta = \frac{x'}{1} \) or \( \frac{x}{r} \) and \( \sin \theta = \frac{y'}{1} \) or \( \frac{y}{r} \).
The ratios $\frac{x}{r}$ and $\frac{y}{r}$ do not depend on the choice of $P$. They depend only on the measure of $\theta$ and thus, are the basic trigonometric functions of $\theta$.

### Sine and Cosine Functions of an Angle in Standard Position

For any angle in standard position with measure $\theta$, a point $P(x, y)$ on its terminal side, and $r = \sqrt{x^2 + y^2}$, the sine and cosine functions of $\theta$ are as follows.

$$\sin \theta = \frac{y}{r} \quad \cos \theta = \frac{x}{r}$$

#### Example 2

Find the values of the sine and cosine functions of an angle in standard position with measure $\theta$ if the point with coordinates $(3, 4)$ lies on its terminal side.

You know that $x = 3$ and $y = 4$. You need to find $r$.

$$r = \sqrt{x^2 + y^2} = \sqrt{3^2 + 4^2} = 5$$

Then write the sine and cosine ratios.

$$\sin \theta = \frac{y}{r} \text{ or } \frac{4}{5} \quad \cos \theta = \frac{x}{r} \text{ or } \frac{3}{5}$$

#### Example 3

Find $\sin \theta$ when $\cos \theta = \frac{5}{13}$ and the terminal side of $\theta$ is in the first quadrant.

Since $\cos \theta = \frac{x}{r} = \frac{5}{13}$ and $r$ is always positive, $r = 13$ and $x = 5$.

Now, find $y$.

$$r = \sqrt{x^2 + y^2} = \sqrt{5^2 + y^2} = 13$$

$$169 = 25 + y^2$$

$$144 = y^2$$

$$y = \pm 12$$

Since $\theta$ is in the first quadrant, $y$ must be positive.

Thus, $\sin \theta = \frac{y}{r} \text{ or } \frac{12}{13}$.
In addition to the ratios \( \frac{y}{r} \) and \( \frac{x}{r} \) that are used to define the sine and cosine functions, four other ratios may be formed using \( x, y, \) and \( r \). These are the tangent, cotangent, secant, and cosecant functions, which are abbreviated \( \tan, \cot, \sec, \) and \( \csc \), respectively. These ratios depend only on the measure of \( \theta \) and thus, provide additional trigonometric functions.

<table>
<thead>
<tr>
<th>Trigonometric Functions of an Angle in Standard Position</th>
</tr>
</thead>
<tbody>
<tr>
<td>For any angle in standard position with measure ( \theta ), a point ( P(x, y) ) on its terminal side, and ( r = \sqrt{x^2 + y^2} ), the trigonometric functions of ( \theta ) are as follows.</td>
</tr>
</tbody>
</table>
| \[
\begin{align*}
\sin \theta &= \frac{y}{r} & \cos \theta &= \frac{x}{r} & \tan \theta &= \frac{y}{x} \\
\csc \theta &= \frac{r}{y} & \sec \theta &= \frac{r}{x} & \cot \theta &= \frac{x}{y}
\end{align*}
\] |

We can write the tangent, cosecant, secant, and cotangent functions in terms of the sine, cosine, and tangent.

\[
\begin{align*}
\tan \theta &= \frac{\sin \theta}{\cos \theta} & \csc \theta &= \frac{1}{\sin \theta} & \sec \theta &= \frac{1}{\cos \theta} & \cot \theta &= \frac{1}{\tan \theta}
\end{align*}
\]

Thus, \( \sin \theta \) and \( \csc \theta \) are reciprocals, as are \( \sec \theta \) and \( \cos \theta \), and \( \tan \theta \) and \( \cot \theta \).

**Example 4**

The terminal side of an angle \( \theta \) in standard position contains the point with coordinates \( (8, -15) \). Find \( \tan \theta \), \( \cot \theta \), \( \sec \theta \), and \( \csc \theta \).

You know that \( x = 8 \) and \( y = -15 \). You need to find \( r \).

\[
r = \sqrt{x^2 + y^2} = \sqrt{8^2 + (-15)^2} = \sqrt{289} = 17
\]

Now, write the ratios.

\[
\begin{align*}
\tan \theta &= \frac{y}{x} \text{ or } \frac{-15}{8} & \cot \theta &= \frac{x}{y} \text{ or } \frac{8}{-15} \\
\sec \theta &= \frac{r}{x} \text{ or } \frac{17}{8} & \csc \theta &= \frac{r}{y} \text{ or } \frac{17}{-15}
\end{align*}
\]

If you know the value of one of the trigonometric functions and the quadrant in which the terminal side of \( \theta \) lies, you can find the values of the remaining five functions.
Example 5

If \( \csc \theta = -2 \) and \( \theta \) lies in Quadrant III, find \( \sin \theta, \cos \theta, \tan \theta, \cot \theta, \) and \( \sec \theta. \)

Since \( \csc \theta \) and \( \sin \theta \) are reciprocals, \( \sin \theta = -\frac{1}{2}. \)

To find the other function values, you must find the coordinates of a point on the terminal side of \( \theta. \) Since \( \sin \theta = -\frac{1}{2} \) and \( r \) is always positive, let \( r = 2 \) and let \( y = -1. \) Find \( x. \)

\[
\begin{align*}
x^2 + y^2 &= r^2 \\
x^2 + (-1)^2 &= 2^2 \\
x^2 &= 3 \\
\pm \sqrt{3} &= x
\end{align*}
\]

Since the terminal side of \( \theta \) lies in Quadrant III, \( x = -\sqrt{3}. \)

Now, write the ratios.

\[
\begin{align*}
\cos \theta &= \frac{-\sqrt{3}}{2} \text{ or } \frac{-\sqrt{3}}{2} \\
\tan \theta &= \frac{-1}{-\sqrt{3}} \text{ or } \frac{\sqrt{3}}{3} \\
\sec \theta &= \frac{2}{-\sqrt{3}} \text{ or } \frac{-2\sqrt{3}}{3} \\
\cot \theta &= \frac{-\sqrt{3}}{-1} \text{ or } \sqrt{3}
\end{align*}
\]

As illustrated in Examples 4 and 5, the values of the trigonometric functions may be either positive, negative, or 0. Since \( r \) is always positive, the signs of the functions are determined by the signs of \( x \) and \( y. \)

CHECKING FOR UNDERSTANDING

Communicating Mathematics

Read and study the lesson to answer each question.

1. Explain why the \( \sin a^\circ \) and \( \sin(a^\circ + 360k^\circ) \), where \( k \) is an integer, are equal. Use an example as part of your explanation.

2. State the quadrant or quadrants in which \( \sin \theta \) and \( \cos \theta \) are both positive.

3. Show that as the measure of \( \theta \) increases from \( 0^\circ \) to \( 90^\circ \), the value of \( \cos \theta \) decreases from 1 to 0.

4. Tell which is greater, \( \sin 12^\circ \) or \( \sin 13^\circ. \)
5-4

Trigonometric Functions of Special Angles

Objectives
After studying this lesson, you should be able to:

- find exact values for the six trigonometric functions of special angles, and
- find decimal approximations for the values of the six trigonometric functions of any angle.

Application
You see an object because light is reflected from the object into your eyes. However, light travels faster in air than it does in water. This can be a bit confusing if you are about to step on a submerged rock while crossing a stream. The rock is not exactly where it appears to be. The displacement of the light ray depends on the angle at which the light strikes the surface of the water from below, \( A \), the depth of the rock, \( t \), and the angle at which the light leaves the surface of the water, \( B \). The measure of displacement, \( x \), is given by the formula below.

\[
x = t \left( \frac{\sin (B - A)}{\cos A} \right)
\]

Find the displacement if \( t \) measures 10 centimeters, the measure of angle \( A \) is 30°, and the measure of angle \( B \) is 42°. This problem will be solved in Example 4.

Before you find the values of the trigonometric functions in the formula above, it is helpful to study the values of trigonometric functions of special angles. One such group of angles are quadrantal angles. It is easy to find the trigonometric functions for quadrantal angles since their terminal sides lie along an axis. Using a unit circle, let \((x, y)\) be the coordinates of the point of intersection of the circle with the terminal side of the angle. For example, \( \cos 180° = -1 \) and \( \sin 180° = 0 \).
Example 1  Find the values of the six trigonometric functions for an angle in standard position that measures $90^\circ$.

$$\sin 90^\circ = 1 \quad \cos 90^\circ = 0$$

$$\tan 90^\circ \text{ is undefined because division by zero is undefined.}$$

$$\cot 90^\circ = 0 \quad \csc 90^\circ = 1$$

$$\sec 90^\circ \text{ is undefined because division by zero is undefined.}$$

The domain of the sine and cosine functions is the set of real numbers, since $\sin \theta$ and $\cos \theta$ are defined for any angle $\theta$. The range of the sine and cosine functions is the set of real numbers between $-1$ and $1$ inclusive, since $(\sin \theta, \cos \theta)$ are the coordinates of points on the unit circle. Since division by zero is undefined, there are several angle measures that are excluded from the domain of the tangent, cotangent, secant, and cosecant functions. The table below summarizes the values of the trigonometric functions of common quadrant angles. The dashes represent undefined values.

<table>
<thead>
<tr>
<th>Angle</th>
<th>$\sin$</th>
<th>$\cos$</th>
<th>$\tan$</th>
<th>$\csc$</th>
<th>$\sec$</th>
<th>$\cot$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0^\circ$</td>
<td>$0$</td>
<td>$1$</td>
<td>$0$</td>
<td>$-1$</td>
<td>$1$</td>
<td>$-1$</td>
</tr>
<tr>
<td>$90^\circ$ or $\frac{\pi}{2}$</td>
<td>$1$</td>
<td>$0$</td>
<td>$-1$</td>
<td>$-1$</td>
<td>$0$</td>
<td>$0$</td>
</tr>
<tr>
<td>$180^\circ$ or $\pi$</td>
<td>$0$</td>
<td>$-1$</td>
<td>$0$</td>
<td>$-1$</td>
<td>$-1$</td>
<td>$-1$</td>
</tr>
<tr>
<td>$270^\circ$ or $\frac{3\pi}{2}$</td>
<td>$-1$</td>
<td>$0$</td>
<td>$1$</td>
<td>$1$</td>
<td>$0$</td>
<td>$0$</td>
</tr>
<tr>
<td>$360^\circ$ or $2\pi$</td>
<td>$0$</td>
<td>$1$</td>
<td>$0$</td>
<td>$1$</td>
<td>$1$</td>
<td>$1$</td>
</tr>
</tbody>
</table>

You can find the trigonometric functions of other special angles by using relationships from geometry. Recall that in a $30^\circ$–$60^\circ$ right triangle, the lengths of the sides are in the ratio $1 : \sqrt{3} : 2$. In a $45^\circ$–$45^\circ$ right triangle, the lengths of the sides are in the ratio $1 : 1 : \sqrt{2}$.

Example 2  Find $\sin 60^\circ$, $\cos 60^\circ$, and $\tan 60^\circ$.

Sketch a $30^\circ$–$60^\circ$ right triangle so that the $60^\circ$ angle is in standard position.

Choose $P(x, y)$ on the terminal side of the angle so that $r = 2$. It follows that $x = 1$ and $y = \sqrt{3}$.

$$\sin 60^\circ = \frac{\sqrt{3}}{2}$$

$$\cos 60^\circ = \frac{1}{2}$$

$$\tan 60^\circ = \sqrt{3}$$
The values found in Example 2 are also those for angles of radian measure \( \frac{\pi}{3} \), since \( 60^\circ = \frac{\pi}{3} \). The chart below summarizes the sine and cosine values for selected angles from 0 to \( \pi \). It is a good idea to memorize these values since you will use them frequently.

<table>
<thead>
<tr>
<th>( \theta ) (in radians)</th>
<th>0</th>
<th>( \frac{\pi}{6} )</th>
<th>( \frac{\pi}{4} )</th>
<th>( \frac{\pi}{3} )</th>
<th>( \frac{\pi}{2} )</th>
<th>( 2\frac{\pi}{3} )</th>
<th>( 3\frac{\pi}{4} )</th>
<th>( \frac{5\pi}{6} )</th>
<th>( \pi )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \theta ) (in degrees)</td>
<td>0</td>
<td>30°</td>
<td>45°</td>
<td>60°</td>
<td>90°</td>
<td>120°</td>
<td>135°</td>
<td>150°</td>
<td>180°</td>
</tr>
<tr>
<td>( \cos \theta )</td>
<td>1</td>
<td>( \frac{\sqrt{3}}{2} )</td>
<td>( \frac{\sqrt{2}}{2} )</td>
<td>( \frac{1}{2} )</td>
<td>0</td>
<td>( -\frac{1}{2} )</td>
<td>( -\frac{\sqrt{2}}{2} )</td>
<td>( -\frac{\sqrt{3}}{2} )</td>
<td>-1</td>
</tr>
<tr>
<td>( \sin \theta )</td>
<td>0</td>
<td>( \frac{1}{2} )</td>
<td>( \frac{\sqrt{2}}{2} )</td>
<td>( \frac{\sqrt{3}}{2} )</td>
<td>1</td>
<td>( \frac{\sqrt{3}}{2} )</td>
<td>( \frac{\sqrt{2}}{2} )</td>
<td>( \frac{1}{2} )</td>
<td>0</td>
</tr>
</tbody>
</table>

You can also use the reference angle for certain angles to find the value of trigonometric functions. Recall that the reference angle is the acute angle formed by the terminal side of a given angle and the \( x \)-axis.

**Example 3**

You may wish to review the reference angle rules in Lesson 5-1.

Find each value.

a. \( \cos \frac{5\pi}{6} \)

b. \( \tan \left( -\frac{11\pi}{4} \right) \)

a. \( \alpha = \frac{5\pi}{6} \), so its terminal side is in Quadrant II.

\[
\alpha' = \pi - \alpha \quad \text{Reference rule b}
\]

\[
= \pi - \frac{5\pi}{6}
\]

\[
= \frac{\pi}{6}
\]

\( \frac{\pi}{6} = 30^\circ \)

\[
\cos \frac{5\pi}{6} = -\cos \frac{\pi}{6} \quad \cos \alpha < 0 \text{ in Quadrant II}
\]

\[
= -\frac{\sqrt{3}}{2}
\]

b. \( -\frac{11\pi}{4} \) is coterminal with \( \frac{5\pi}{4} \) in Quadrant III.

\[
\alpha' = \alpha - \pi \quad \text{Reference rule c}
\]

\[
= \frac{5\pi}{4} - \pi
\]

\[
= \frac{\pi}{4}
\]

\( \frac{\pi}{4} = 45^\circ \)
\[
\tan \left( -\frac{11\pi}{4} \right) = \tan \left( \frac{-\pi}{4} \right) = 1
\]

\text{c. } \csc \frac{29\pi}{3}
\cfrac{29\pi}{3} \text{ is coterminal with } \frac{5\pi}{3} \text{ in Quadrant IV.}
\alpha' = 2\pi - \alpha \hspace{1cm} \text{Reference rule d}
\begin{align*}
\alpha' &= 2\pi - \frac{5\pi}{3} \\
&= \frac{\pi}{3}
\end{align*}
\csc \frac{29\pi}{3} = -\csc \frac{\pi}{3} \hspace{1cm} \csc < 0 \text{ in Quadrant IV}
\begin{align*}
&= -\csc \frac{\pi}{3} \\
&= -\frac{1}{\sin \frac{\pi}{3}} \\
&= -\frac{2}{\sqrt{3}} \text{ or } -\frac{2\sqrt{3}}{3}
\end{align*}

The values of the trigonometric functions of any angle can be approximated using a scientific calculator. Most approximate values are given to four decimal places. Always check to see whether the calculator is in radian or degree mode.

Example 4

Refer to the application at the beginning of the lesson. Find the displacement if \( t \) measures 10 cm, the measure of angle \( A \) is 30°, and the measure of angle \( B \) is 42°.

\[
X = t \left( \frac{\sin (B - A)}{\cos A} \right)
\]
\[
= 10 \left( \frac{\sin (42° - 30°)}{\cos 30°} \right)
\]
\[
= 10 \left( \frac{\sin 12°}{\cos 30°} \right) \hspace{1cm} \text{Use your calculator. Make sure it is in degree mode.}
\]

10 \( \times \) 12 \( \sin \) 30° \( \cos \) 12° \( \approx \) 2.407757

The rock submerged 10 centimeters under water is actually about 2.4 centimeters from where it appears.
5-5 Right Triangles

Objective

After studying this lesson, you should be able to:

- solve right triangles.

Application

The longest truck-mounted ladder used by the Dallas Fire Department is 108 feet long and consists of four hydraulic sections. Gerald Travis, aerial expert for the department, indicates that the optimum operating angle of this ladder is 60°. Outriggers, with an 18-foot span between each, are used to stabilize the ladder truck and permit operating angles greater than 60°, allowing the ladder truck to be closer to buildings in the downtown streets of Dallas. Assuming the ladder is mounted 8 feet off the ground, how far from an 84-foot burning building should the base of the ladder be placed to achieve the optimum operating angle of 60°? How far should the ladder be extended to reach the roof? This problem will be solved in Example 4.

Right triangles can be used to define trigonometric functions. Let A, B, and C designate the vertices of a right triangle and the angles at those vertices. The measures of the sides opposite angles A, B, and C are designated by a, b, and c, respectively. All right triangles having acute angles congruent to angles A and B are similar. Thus, the ratios of corresponding sides are equal. These ratios are determined by the measures of the acute angles. Therefore, any two congruent angles of different right triangles will have the same ratios associated with them.

For an acute angle A in right triangle ABC, the trigonometric functions are as follows.

\[
\begin{align*}
\sin A &= \frac{\text{side opposite}}{\text{hypotenuse}} = \frac{a}{c} \\
\cos A &= \frac{\text{side adjacent}}{\text{hypotenuse}} = \frac{b}{c} \\
\tan A &= \frac{\text{side opposite}}{\text{side adjacent}} = \frac{a}{b} \\
\csc A &= \frac{\text{hypotenuse}}{\text{side opposite}} = \frac{c}{a} \\
\sec A &= \frac{\text{hypotenuse}}{\text{side adjacent}} = \frac{c}{b} \\
\cot A &= \frac{\text{side adjacent}}{\text{side opposite}} = \frac{b}{a}
\end{align*}
\]

SOH-CAH-TOA is a mnemonic device commonly used for remembering the first three equations.

\[
\begin{align*}
\sin \theta &= \frac{\text{Opposite}}{\text{Hypotenuse}} \\
\cos \theta &= \frac{\text{Adjacent}}{\text{Hypotenuse}} \\
\tan \theta &= \frac{\text{Opposite}}{\text{Adjacent}}
\end{align*}
\]
6-1  Graphs of the Trigonometric Functions

Objective
After studying this lesson, you should be able to:
• use the graphs of the trigonometric functions.

Application

We live in an electronic age. Electricity powers our lights, heats our homes, and even runs our cars. However, unlike light or sound, under normal circumstances we cannot sense this form of energy directly. We must use tools, like an oscilloscope, to measure the behavior of electric currents.

The electric power that is used in American homes is delivered in alternating current. An alternating current surges in one direction and then the other in a way that can be represented on the oscilloscope by a sine curve.

You could generate the coordinates of points on the parent graphs of trigonometric functions with a calculator or spreadsheet program. The graphs may be generated by using a graphing calculator or graphing software. The parent graphs of the six basic trigonometric functions are shown below and on the next page.

\[ y = \sin \theta \]

\[ y = \cos \theta \]
Do you notice a similarity between the sine and cosine curves? The cosine curve is the sine curve translated 90° to the left along the \( \theta \)-axis. Since the cosecant and secant ratios are the reciprocals of the sine and cosine ratios, the same relationship holds true for their curves. However, this does not hold true for the tangent and cotangent curves. Why?

A quadrantal angle is an angle in standard position whose terminal side coincides with one of the coordinate axes. So the measures of the quadrantal angles are \( \cdots, -270^\circ, -180^\circ, -90^\circ, 0^\circ, 90^\circ, 180^\circ, 270^\circ, \cdots \). Knowing the characteristics of the basic trigonometric graphs can help you quickly determine quadrantal values of the trigonometric functions. The value of the sine function is 0 at \(-360^\circ, -180^\circ, 0^\circ, 180^\circ, 360^\circ\), and at all other integral multiples of 180°. The maximum value of the sine function is 1 for 90° or -270°, and the minimum is -1 for 270° or -90°. The fact that the trigonometric functions are periodic allows us to find more function values easily.

### Periodic Function and Period

A function is periodic if, for some real number \( \alpha \), \( f(x + \alpha) = f(x) \) for each \( x \) in the domain of \( f \). The least positive value of \( \alpha \) for which \( f(x) = f(x + \alpha) \) is the period of the function.
Since the sine function is periodic, its graph repeats itself every $360^\circ$. So we can find other zero, maximum, and minimum values easily. For example, since we know that $\sin 90^\circ = 1$, $\sin (90^\circ + 360k^\circ) = 1$ for any integral value of $k$. The same methods can be used to find special values of the other functions.

**Example 1**

**Use the graph of the cosine function to find the values of $\theta$ for which $\cos \theta = 1$.**

When $\cos \theta = 1$ and $-360^\circ \leq \theta \leq 360^\circ$, the value of $\theta$ is $360^\circ$, $0^\circ$, or $360^\circ$. Since the cosine function has a period of $360^\circ$, the values of $\theta$ for which $\cos \theta = 1$ are given by $0^\circ + 360k^\circ$ or simply $360k^\circ$, where $k$ is any integer.

The graph of a trigonometric function may be drawn from the knowledge of its shape and the values of the function at integral multiples of $90^\circ$.

**Example 2**

**Graph the sine curve in the interval $-540^\circ \leq \theta \leq 0^\circ$.**

Find the value of sine $\theta$ for $\theta = -540^\circ$, $-450^\circ$, $-270^\circ$, $-180^\circ$, $-90^\circ$, and $0^\circ$.

Plot the points from these ordered pairs.

$(-540^\circ, 0), (-450^\circ, -1), (-360^\circ, 0), (-270^\circ, 1), (-180^\circ, 0), (-90^\circ, 1), (0^\circ, 0)$

Connect these points with a smooth continuous curve.

---

**CHECKING FOR UNDERSTANDING**

**Communicating Mathematics**

Read and study the lesson to answer each question.

1. **Describe** how a cosine curve can be translated to coincide with the graph of a sine curve.

2. The vertical lines in the graphs of the tangent, cotangent, secant, and cosecant functions are asymptotes. What do they indicate about the values of $\theta$ where the asymptotes cross the $\theta$-axis?