❖ Using repeated addition

Often the first strategy students use to solve multiplication problems is repeated addition. This is because they are viewing the situation additively, rather than multiplicatively. To solve 6×4 (how many oranges are in this one-layer box), students will write (or think about) $4 + 4 + 4 + 4 + 4 + 4$ and then add to find the total. Repeated addition should be seen as a starting place in the journey, but not as the end point. As you confer with students, you will need to help them to keep track of the groups, and you can encourage more efficient grouping (such as turning the 6 fours into pairs of fours, resulting in 3 eights).

❖ Skip-counting

The struggle to keep track of the groups usually pushes students to skip-count. Although they are still thinking about the situation additively rather than multiplicatively, they keep track of the groups mentally and skip-count. To figure out how many oranges are in a 6×4 one-layer box, they might say 4, 8, 12, 16, 20, 24.

❖ Using partial products

An important shift in multiplicative thinking occurs when students begin to make partial products—they use a fact they know to make another. For example, they might reason that a 3×4 box of oranges with 2 layers holds $(3 \times 4) + (3 \times 4)$ oranges. Or they might multiply 12×18 by using $(10 \times 18) + (2 \times 18)$. The big idea underlying this strategy is the distributive property.

❖ Using ten-times

Once students begin to make use of partial products, an important strategy to encourage is the use of the ten-times partial product. This can be very helpful for dealing with larger numbers—for example, 12×30. It is helpful to think about this as $12 \times 3 \times 10$. Of course, this strategy is helpful only if students have constructed an understanding of the place value patterns that occur when multiplying by the base.

❖ Doubling and halving

As students' multiplicative reasoning becomes stronger, they develop the ability to group more efficiently. They begin to realize that if they double the number of groups and want the total product to be the same, they need to halve the amount in each group: $4 \times 6 = 8 \times 3$.

❖ Factoring and grouping flexibly

Doubling and halving can be generalized to tripling and thirding, or quadrupling and quartering, etc. The big idea underlying the reason these strategies work is the associative property of multiplication. In multiplication, the factors can be associated in a variety of ways that may make multiplication problems easy to do mentally. For example, $28 \times 5 = (14 \times 2) \times 5 = 14 \times (2 \times 5) = 140$. Efficient computation results when students realize that they can factor the factors and regroup them to make the computation easier.

MATHEMATICAL MODELING

The primary focus of this unit is the extension of a two-dimensional array model to the three-dimensional array model for multiplication. Initially the model is introduced as the arrangement of oranges in a box with multiple layers. Boxes are used to explore the associative property and the relationship of surface area to volume. Eventually cubic box units are used to generalize a formula for determining the volume of rectangular prisms.

The array is a powerful model for multiplicative thinking because it can support the development of the following:

- a wide range of strategies (skip-counting, repeated addition, doubling, doubling and halving, partial products) and big ideas like the distributive, associative, and commutative properties
- visual representations for what multiplication means (e.g., 6×8 can be understood as 6 rows of 8 squares or 8 columns of 6 squares)
- an understanding of area and perimeter, surface area, dimensions, and volume

Models go through three stages of development (Gravemeijer 1999; Fosnot and Dolk 2002):

❖ ***model of the situation***

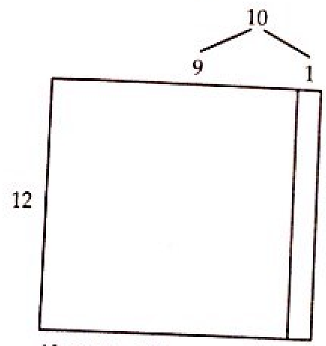
❖ ***model of students' strategies***

❖ ***model as a tool for thinking***

introduced as an arrangement (rows and columns) of truffles in a box. As students explore and design boxes for the truffles, the blueprint (graph paper arrays) emerges as a representation of the box—and now it is a two-dimensional rectangular array. Quick images are introduced using the 2 × 5 and the 1 × 5 blueprints as units, and these arrays depicting the situation become the scaffold that allows students to envision how larger arrays can be made with smaller arrays. Eventually, outlines of the boxes are used in which there are no rows and columns to count.

❖ *Model of students' strategies*

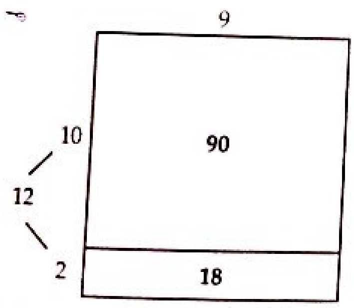
Students benefit from seeing the teacher model their strategies. Once a model has been introduced as a representation of the situation, you can use it to display student strategies. If a student says about solving 12 × 9, "I used 12 × 10 and subtracted 12," draw the following:



12 × 10 = 120

12 × 9 = (12 × 10) − (12 × 1) = 120 − 12 = 108

Another student in calculating 12 × 9 might say, "I knew 10 × 9 and 2 × 9 and I just added them together to get 108." Here you can represent the partial products as follows:



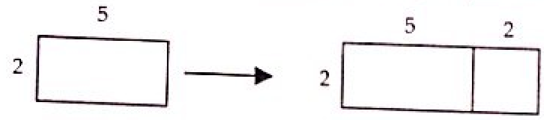
12 × 9 = (10 × 9) + (2 × 9) = 90 + 18 = 108

Representations like these give students a chance to see and discuss each other's strategies.

❖ *Model as a tool for thinking*

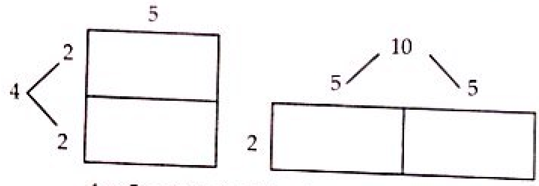
Eventually, students become able to use the model as a tool to think with—to explore and prove their ideas about multiplicative reasoning. Here, as the dimensions of a box are visualized in relationship to the dimensions of other boxes, students can explore the distributive property of multiplication over addition (or subtraction), the associative property, and the commutative property.

1. The distributive property of multiplication over addition: the original 2 × 5 open array is used to figure out the dimensions of a larger open array:



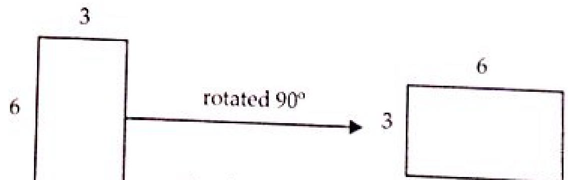
(2 × 5) + (2 × 2) = 2 × 7

2. Doubling and halving can be examined and then the array can be used as a tool to explore its generalization to the associative property:



4 × 5 = (2 × 2) × 5 = 2 × (2 × 5) = 20

3. The commutative property, equivalence, and congruence:



6 × 3 = 3 × 6

Many opportunities to discuss these landmarks in mathematical development will arise as you work through this unit. Look for moments of puzzlement. Don't hesitate to let students discuss their ideas and check and recheck their strategies. Celebrate their accomplishments! They are young mathematicians at work!

A graphic of the full landscape of learning for multiplication and division is provided on page 11. The purpose of the graphic is to allow you to see the longer journey of students' mathematical